CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar, Friday conversations:,

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at “free formulas” and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimer. — The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[23] On the rôle of local analysis in the computation of polylogarithms and harmonic sums.

1. In the preceding weeks, we have considered the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w\in X^*} w \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (1)$$

2. This identity, formulated with a basis of Lie polynomials and its dual holds true, not only for other bases but also with other Lie algebras (precisely those that are free as **k**-modules).
3. At first, one must pass from a basis of the Lie algebra in question $\mathfrak{g}$ (if it exists) to a basis of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then, one exploits the factorials due to the comultiplication is order to get the infinite product.
4. Today we will see how to extend the indexation of Polylogarithmic functions and Harmonic sums.

Introduction.

The aim of this talk is to explain how to extend polylogarithms

$$\mathrm{Li}(s_1,\ldots s_r) = \sum_{n_1>n_2>\ldots n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \text{ for } |z|<1 \tag{2}$$

They were a priori coded by lists $(s_1,\ldots s_r)$ but, when $s_i \in \mathbb{N}_+$, they admit an *iterated integral representation* and are better coded by words with letters in $X=\{x_0,x_1\}$. We will use the one-to-one correspondences.

$$({\color{red}s_1},\ldots,{\color{red}s_r}) \in \mathbb{N}_+^r \leftrightarrow x_0^{{\color{red}s_1}-1}x_1\ldots x_0^{{\color{red}s_r}-1}x_1 \in X^*x_1 \leftrightarrow y_{\color{red}s_1}\ldots y_{\color{red}s_r} \in Y^* \tag{3}$$

- $\mathrm{Li}(s)[z]$ is Jonquière and, for $\Re(s)>1$, one has $\mathrm{Li}(s)[1]=\zeta(s)$
- Completed by $Li(x_0^n) = \frac{log^n(z)}{n!}$ this provides a family of $\mathbb{C}$-independant functions (linearly) admitting an analytic continuation on the cleft plane $\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$ or $\widetilde{\mathbb{C}\setminus\{0,1\}}$.

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Introduction: Review of the facts.

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ $(\Re(s) > 1)$
- when one multiplies two of these, one gets quantities like

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1,n_2\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1,s_2) + \zeta(s_1+s_2) + \zeta(s_2,s_1)$$

- and, with several of them, we are led to the following definition of MultiZeta Values (MZV), converging in
$$\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r \,|\, \forall m = 1,\ldots,r, \Re(s_1) + \ldots + \Re(s_m) > m\} .$$

$$\zeta(s_1,\ldots,s_k) := \sum_{n_1>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}} \tag{4}$$

- On the other hand, one has the classical polylogarithms defined, for $k \geq 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n\geq 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n\geq 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

Introduction: Review of the facts/2

- The analogue of the classical polylogarithms for MZV reads

$$Li_{y_{s_1}\ldots y_{s_k}}(z) := \sum_{n_1>\ldots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}} \ ; \ |z|<1$$

- They satisfy the recursion (ladder stepdown)

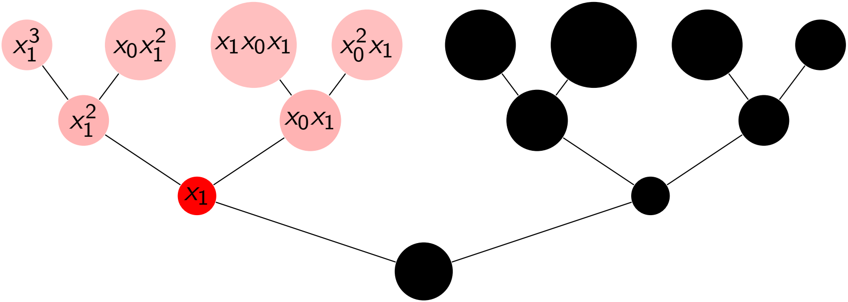
$$\begin{aligned} z\frac{d}{dz}Li_{y_{s_1}\ldots y_{s_k}} &= Li_{y_{s_1-1}\ldots y_{s_k}} \text{ if } s_1>1 \\ (1-z)\frac{d}{dz}Li_{y_1 y_{s_2}\ldots y_{s_k}} &= Li_{y_{s_2}\ldots y_{s_k}} \text{ if } k>1 \end{aligned} \tag{5}$$

which, with $s_i \in \mathbb{N}_{\geq 1},\ k\geq 1$, ends at the "seed"

$$\mathrm{Li}_{y_1}(z) = \mathrm{Li}_1(z) = \log(\frac{1}{1-z}) \tag{6}$$

- For the next step, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z \frac{f(s)}{s}ds$ (resp. $\int_0^z \frac{f(s)}{1-s}ds$) - with x_0 (resp. x_1).

Tree of outputs (so far).



Some coefficients with $X = \{x_0, x_1\};\ u_0(z) = \frac{1}{z};\ u_1(z) = \frac{1}{1-z},\ *_0 = 0$

$$\langle S \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S \mid x_0x_1 \rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$$

$$\langle S \mid x_0^2x_1 \rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n^3} \quad ; \quad \langle S \mid x_1x_0x_1 \rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1} \frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S \mid x_0x_1^2 \rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1} \frac{z^{n_1}}{n_1^2n_2}$$

; above "cl. not." stands for "classical notation"

Introduction: Review of the facts/3

- Calling S the prospective generating series

$$S = \sum_{w \in X^*} \underbrace{\langle S \mid w \rangle}_{\in \mathcal{H}(\Omega)} w \ ; \ X = \{x_0, x_1\} \tag{7}$$

V. Drinfel'd [1] indirectly proposed a way to complete the tree:

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S & (NCDE) \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \tag{8}$$

from the general theory, this system has a unique solution which is precisely Li (called G_0 in [1]) ; $S \mapsto \mathbf{d}(S)$ being the term by term derivation of the coefficients.

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals (see also [13]).

Explicit construction of Drinfeld's G_0.

Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

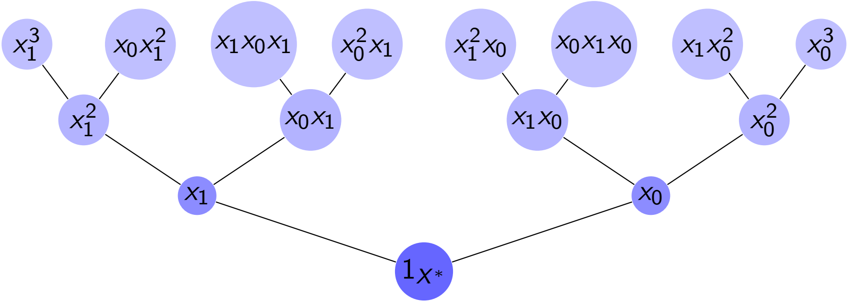
$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \ (w \in x_0^*) \\ \int_0^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ (w \in x_0 X^* x_1 x_0^*) \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_0 in [1]).

Complete tree of outputs.



As an example, we compute some coefficients

$$\langle \mathrm{Li} \mid x_0^n \rangle = \frac{log(z)^n}{n!} \quad ; \quad \langle \mathrm{Li} \mid x_1^n \rangle = \frac{(-log(1-z))^n}{n!}$$

$$\langle \mathrm{Li} \mid x_0x_1 \rangle = \mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = \langle \mathrm{Li} \mid x_1 \sqcup\!\sqcup\, x_0 - x_0x_1 \rangle(z)$$

$$\langle \mathrm{Li} \mid x_0^2x_1 \rangle = \mathrm{Li}_3(z) = \sum_{n\geq 1} \frac{z^n}{n^3} \quad ; \quad \langle \mathrm{Li} \mid x_1x_0 \rangle = (-log(1-z))log(z) - \mathrm{Li}_2(z)$$

$$\langle \mathrm{Li} \mid x_0^{r-1}x_1 \rangle = \mathrm{Li}_r(z) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad ; \quad \langle \mathrm{Li} \mid x_1^2x_0 \rangle = \langle \mathrm{Li} \mid \frac{1}{2}(x_1 \sqcup\!\sqcup\, x_1 \sqcup\!\sqcup\, x_0) - (x_1 \sqcup\!\sqcup\, x_0x_1) + x_0x_1^2 \rangle$$

Li From a NCDE.

The generating series $S = \sum_{w \in X^*} Li(w)$ satisfies (and is unique to do so)

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases} \tag{9}$$

with $X = \{x_0, x_1\}$. This is, up to the sign of x_1, the solution G_0 of Drinfel'd [13] for KZ3[a]. We define this unique solution as Li. All Li_w are $\mathbb{C}$- and even $\mathbb{C}(z)$-linearly independant (see CAP 17 *Linear independance without monodromy* [23]).

[a] In fact, the path from KZ3 to these equations is done through a counter-homogenization (see Vu's forthcoming talks).

Domain of Li (global, definition)

In order to extend indexation of Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges unconditionally for compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{10}$$

Starting the ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}^{(1)}_\bullet} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Examples

$$Li_{x_0^*}(z) = z,\quad Li_{x_1^*}(z) = (1-z)^{-1},\ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Main difference between $\alpha_{z_0}^z$ and α_0^z.

5. Here, we still work with
$\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $u_0 = 1/z,\ u_1 = 1/(1-z)$
6. $\alpha_{z_0}^z,\ \alpha_0^z :\ X^* \longrightarrow \mathcal{H}(\Omega)$ are both shuffle characters (see below) but they satisfy different growth conditions.
7. **With $\alpha_{z_0}^z$, ($z_0 \in \Omega$).** — Let us denote $\mathfrak{K}(\Omega)$ the set of compact subsets of Ω. One can show that, for all $K \in \mathfrak{K}(\Omega)$, there exists $M_K > 0$ s.t.
$$(\forall w \in X^+)(\ ||\langle \alpha_{z_0}^z \mid w\rangle||_K \leq M_K \frac{1}{(|w|-1)!}\ ) \qquad (11)$$
8. This entails that, given a rational series $T = \sum_{n\geq 0} T_n$ (where $T_n = \sum_{|w|=n} \langle T \mid w\rangle$), the series, for all $K \in \mathfrak{K}(\Omega)$
$$\sum_{n\geq 0} ||\langle \alpha_{z_0}^z \mid T_n\rangle||_K < +\infty$$
9. We will say that $T \in Dom(\alpha_{z_0}^z)$ and set $\alpha_{z_0}^z(T) = \sum_{n\geq 0} \langle \alpha_{z_0}^z \mid T_n\rangle$.

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/2$

10. In fact, α_0^z satisfies no condition of the type (11) because, with $x_0^* x_1$ (Jonquière branch), we can see that
    1. for $n \geq 1$, $(x_0^* x_1)_n = x_0^{n-1} x_1$, then
    $$\langle \mathrm{Li}(z) \mid x_0^{n-1} x_1 \rangle = \langle \alpha_{z_0}^z \mid x_0^{n-1} x_1 \rangle = J_n(z) = \sum_{k \geq 1} \frac{z^k}{k^n} \qquad (12)$$
    2. The series $\sum_{n \geq 0} J_n$ does not converge (even pointwise) on $]0,1[$ because,
    $$x \in]0,1[\Longrightarrow J_n(x) \geq x$$
    3. So, what can be salvaged ? $\rightarrow$ in fact, conditions (growth or other) implying absolute convergence at the level of words is hopeless because of restriction and we would like to preserve
    $$\mathrm{Li}(x_0^*) = z \ ; \ \mathrm{Li}(x_1^*) = 1/(1-z) \ ; \ \mathrm{Li}(S ⧢ T) = \mathrm{Li}(S).\mathrm{Li}(T) \qquad (13)$$
    and then $\mathrm{Li}\left((x_0 + x_1)^*\right) = z/(1-z)$

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/3$

11. Then, we must have a criterium (for admitting a series in $Dom(\mathrm{Li})$)
12. Fortunately $\mathcal{H}(\Omega)$ shares with finite dimensional spaces the following property

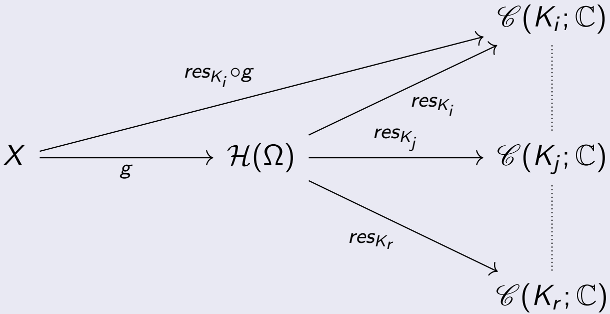
$$\text{Unconditional convergence} \Longleftrightarrow \text{Absolute convergence} \tag{14}$$

13. **Unconditional convergence** for a series $\sum_{n\geq 0} u_n$ means convergence "independent of the order" i.e. that $\sum_{n\geq 0} u_{\sigma(n)}$ converges whatever $\sigma \in \mathfrak{S}_{\mathbb{N}}$.
14. **Absolute convergence** is wrt the continuous seminorms of the space.
15. Time is ripe now to speak of the standard topology of $\mathcal{H}(\Omega)$.
16. For $K \in \mathfrak{K}(\Omega)$, we introduce the seminorm (norm if Ω is connected and $K^\circ \neq \emptyset$)

$$||f||_K = \sup_{z\in K} |f(z)|$$

Initial topologies.

17 We now use a very very general construction, well suited both for series and holomorphic functions (and many other situations), that of initial topologies (see [33] and, for a detailed construction [6], Ch1 §2.3)

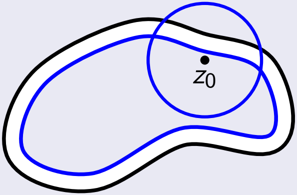


18 So $\mathcal{H}(\Omega)$ is a locally convex TVS whose topology is defined by the family of seminorms $(|| \ ||_K)_{K\in\mathfrak{K}(\Omega)}$.

Topology of $\mathcal{H}(\Omega)$ cont'd.

17. In fact, every $\Omega \subset \mathbb{C}$ is σ-compact, this means that one can construct a sequence $(K_n)_{n\geq 1}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \geq 1)(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\geq 1}\,\mathcal{C}(K_n;\mathbb{C})$ (for the topology on the cube, see a next CCRT).

$$K_n = \{z \in \Omega \mid d(z, z_0) \leq n \text{ and } d(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n}\}.$$



18. We will see more (step-by-step and starting from scratch) on the topology of the cube and separability in the CCRT devoted to convergence questions).

Properties of $\mathcal{H}(\Omega)$ and domain of Li.

17. If $\Omega \neq \emptyset$, $\mathcal{H}(\Omega)$ is not normable because, there are two continuous operators
$$a^\dagger : \ f \mapsto z.f \ ; \ a : \ f \mapsto \frac{d}{dz} f$$
such that $[a, a^\dagger] = Id_{\mathcal{H}(\Omega)}$ (**Hint** Compute $ad_a(e^{ta^\dagger})$).
18. $\mathcal{H}(\Omega)$ has property (14) (nuclearity).
19. This leads us to the following

Definition

Let $T \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$, we define (with $[S]_n := \sum_{|w|=n} \langle S \mid w \rangle\, w$)

$$Dom(T) = \{S \in \mathbb{C}\langle\langle X \rangle\rangle \mid \sum_{n \geq 0} \langle T \mid \ [S]_n \rangle \text{ cv inconditionally}\} \tag{15}$$

If $S \in Dom(T)$, we set $\langle T \mid S \rangle := \sum_{n \geq 0} \langle T \mid \ [S]_n \rangle$.

Shuffle properties and domain of Li.

17. In the case when T is a shuffle character, we have

Theorem (GD, Quoc Huan Ngô, HNM [14] for Li)

Let $T \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ such that

$$\langle T \mid : \ P \mapsto \langle T \mid P \rangle \ (\mathbb{C}\langle X \rangle \to \mathcal{H}(\Omega)) \tag{16}$$

is a shuffle character. then
i) $Dom(T)$ is a shuffle subalgebra of $(\mathbb{C}\langle\langle X \rangle\rangle, ш, 1_{X^*})$.
ii) $\langle T \mid S_1 ш S_2 \rangle = \langle T \mid S_1 \rangle \langle T \mid S_2 \rangle$ i.e. $S \mapsto \langle T \mid S \rangle$ is a shuffle character of $(Dom(T), ш, 1_{X^*})$ that we will still denote $\langle T \mid$.
iii) Then $Im(\langle T \mid\ )$ is a (unital) subalgebra of $\mathcal{H}(\Omega)$.
iv) In particular (see **infra** *for an algebraic proof), $z = \mathrm{Li}(x_0^*)$ and then, $\mathbb{C}[z] \subset Im(Dom(\mathrm{Li}))$.*

Open problems and some solved.

18. Do we have $\mathcal{H}(\Omega) = \overline{Im(Dom(\mathrm{Li}))}$ $(= \overline{Im(\mathrm{Li})})$? (in other words does it exist inaccessible $f \in \mathcal{H}(\Omega)$?)
19. If $z_0 \notin \Omega$, does $1/(z - z_0)$ belong to $Im(\mathrm{Li})$? ($z_0 \in \overline{\Omega}$ and $z_0 \notin \overline{\Omega}$)
20. (Solved) Are there non-rational series in $Dom(\mathrm{Li})$? (answer **yes**)
21. (Solved) Is $\mathbb{C}^{rat}\langle\langle X \rangle\rangle$ contained in $Dom(\mathrm{Li})$ (answer **no**)
22. What is the topological complexity of $Dom(\mathrm{Li})$ in the **Borel hierarchy** (Addison notations, see [24] for details and use the convenient framework of polish spaces [7], ch IX).
23. **Borel hierarchy**: We recall that this hierarchy is indexed by ordinals and defined as follows
    1. A set is in $\mathbf{\Sigma}^0_1$ if and only if it is open.
    2. A set is in $\mathbf{\Pi}^0_\alpha$ if and only if its complement is in $\mathbf{\Sigma}^0_\alpha$.
    3. A set A is in $\mathbf{\Sigma}^0_\alpha$ for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in $\mathbf{\Pi}^0_{\alpha_i}$ for some $\alpha_i < \alpha$ and $A = \bigcup A_i$.
    4. A set is in $\mathbf{\Delta}^0_\alpha$ if and only if it is both in $\mathbf{\Sigma}^0_\alpha$ and in $\mathbf{\Pi}^0_\alpha$.

Open problems and some solved/2

24. From slide (11), one can remark that the iterated integrals are based on two integrators, informally defined as

$$\iota_1(f) := \int_0^z f(s)\frac{ds}{1-s}\ ;\ \iota_0(f) := \int_{z_0}^z f(s)\frac{ds}{s} \text{ with } z_0 \in \{0,1\} \tag{17}$$

ι_1 is defined and continous on $\mathcal{H}(\Omega)$ and ι_0 is defined on $span_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$[a] (context-dependent) and not continuous [14] on this set (see below).
Problem What is the Baire class of ι_0 ?

25. Recall that $\mathfrak{K}(\Omega)$ admits a cofinal sequence $(K_n)_{n\in\mathbb{N}}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \in \mathbb{N})(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\in\mathbb{N}}\mathcal{C}(K_n;\mathbb{C})$.

26. Recall that (see [14] and slide SI.18)

$$K_n = \{z \in \Omega \mid d(z,z_0) \le n \text{ and } d(z,\mathbb{C}\setminus\Omega) \ge \frac{1}{n}\}.$$

[a] It can be a little bit extended, see our paper [14].

Properties of the extended Li.

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and so is
$$Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$$
2. For $S, T \in Dom(Li)$, we have
$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D being the open unit slit disc and $Dom(Li, D)$ defined similarly), whereas $x_0^* x_1 \notin Dom(Li, D)$. Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$. Contrariwise one can show that, for $|t| < 1$,
$$\mathrm{Li}_{(tx_0)^* x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n-t}$$

Passing to harmonic sums $\mathrm{H}_w,\ w \in Y^*$.

Polylogarithms having a removable singularity at zero

The following proposition helps us characterize their indices.

Proposition

Let $f(z) = \langle \mathrm{Li} \mid P \rangle = \sum_{w \in X^*} \langle P \mid w \rangle \mathrm{Li}_w$. The following conditions are equivalent

i) *f can be analytically extended around zero*

ii) $P \in \mathbb{C}\langle X \rangle x_1 \oplus \mathbb{C}.1_{X^*}$

We recall the expansion (for $w \in X^* x_1 \sqcup \{1_{X^*}\},\ |z| < 1$)

$$\frac{\mathrm{Li}_w(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\pi_Y(w)}(N)\, z^N \tag{18}$$

Global and local domains.

This proposition and the lemma lead us to the following definitions.

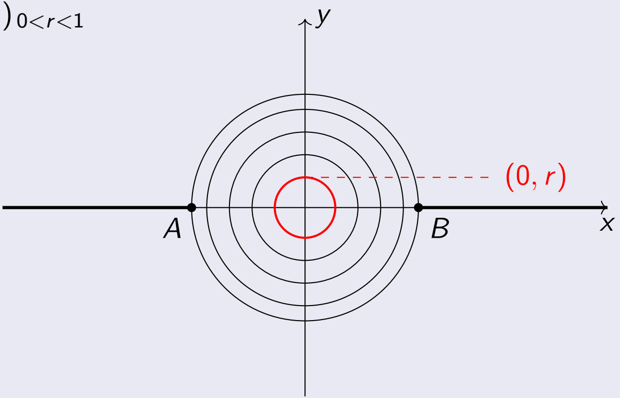
1. *Global domains*.–
   Let $\emptyset \neq \Omega \subset \widetilde{B}$ (with $B = \mathbb{C} \setminus \{0,1\}$), we define $Dom_{\Omega}(Li) \subset \mathbb{C}\langle\langle X \rangle\rangle$ to be the set of series $S = \sum_{n \geq 0} S_n$ (with $S_n = \sum_{|w|=n} \langle S \mid w \rangle\, w$ each homogeneous component) such that $\sum_{n \in \mathbb{N}} Li_{S_n}$ is unconditionally convergent for the compact convergence (UCC) [26].
   As examples, we have Ω_1, the doubly cleft plane then $Dom(\mathrm{Li}) := Dom_{\Omega_1}(\mathrm{Li})$ or $\Omega_2 = \widetilde{B}$

2. *Local domains around zero (fit with* H*-theory)*.–
   Here, we consider series $S \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}\,1_{X^*})$ (i.e. $supp(S) \cap X x_0 = \emptyset$).
   We consider radii $0 < R \leq 1$, the corresponding open discs $D_R = \{z \in \mathbb{C} |\ |z| < R\}$ and define

$$
\begin{aligned}
Dom_R(\mathrm{Li}) &:= \{S = \Sigma_{n \geq 0} S_n \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}1_{\Omega}) | \sum_{n \in \mathbb{N}} Li_{S_n} \text{ (UCC) in } D_R\} \\
Dom_{loc}(\mathrm{Li}) &:= \cup_{0 < R \leq 1} Dom_R(\mathrm{Li}).
\end{aligned}
$$

Local domains.

27. Local domains: the domain of convergence of $\mathrm{Li}_w,\ w \in X^*x_1$ is $\mathbb{C} \setminus (]-\infty,-1] \cup [1,+\infty[)$ and these functions are Taylor expandable around zero. With $S = \sum_{n\geq 0} S_n \in \mathbb{C}\langle\langle X \rangle\rangle$, we study the inconditional convergence of $\sum_{n\geq 0} \mathrm{Li}_{S_n}(z)$ within different open disks $(B_{(0,0)}(r))_{0<r<1}$



Properties of the domains.

Theorem A

1. For all $\emptyset \neq \Omega \subset \widetilde{B}$, $Dom_{\Omega}(\mathrm{Li})$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and so are the $Dom_R(\mathrm{Li})$.
2. $R \mapsto Dom_R(\mathrm{Li})$ is strictly decreasing for $R \in]0,1]$.
3. All $Dom_R(\mathrm{Li})$ and $Dom_{loc}(\mathrm{Li})$ are shuffle subalgebras of $\mathbb{C}\langle\langle X\rangle\rangle$ and $\pi_Y(Dom_{loc}(\mathrm{Li}))$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y\rangle\rangle$.
4. Conversely, let $T(z) = \sum_{N\geq 0} a_N z^N$ be a Taylor series i.e. such that $\limsup_{N\to+\infty} |a_N|^{1/N} = B < +\infty$, then the series

$$S = \sum_{N\geq 0} a_N(-(-x_1)^+)^{ш N} \tag{19}$$

is summable in $\mathbb{C}\langle\langle X\rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1\rangle\rangle$) and $S \in Dom_R(Li)$ with $R = \frac{1}{B+1}$ and $\mathrm{Li}_S = T(z)$.

Theorem A/2

5. Let $S \in Dom_R(\mathrm{Li})$ and $S = \sum_{n\geq 0} S_n$ (homogeneous decomposition), we define[a] $N \mapsto \mathrm{H}_{\pi_Y(S)}(N)$ by

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N \ . \tag{20}$$

Moreover, for all $r \in]0, R[$, we have

$$\sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y(S_n)} r^N| < +\infty, \tag{21}$$

in particular, for all $N \in \mathbb{N}$ the series (of complex numbers) $\sum_{n\geq 0} \mathrm{H}_{\pi_Y(S_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

[a] This definition is compatible with the old one when S is a polynomial.

Theorem A/3

6. Conversely, let $Q \in \mathbb{C}\langle\langle Y \rangle\rangle$ with $Q = \sum_{n\geq 0} Q_n$ (decomposition by weights), we suppose that it exists $r \in]0,1]$ such that

$$\sum_{n,N\geq 0} |\mathrm{H}_{Q_n}(N) r^N| < +\infty \tag{22}$$

in particular, for all $N \in \mathbb{N}$, $\sum_{n\geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ unconditionally.
Under such circumstances, $\pi_X(Q) \in Dom_r(\mathrm{Li})$ and, for all $|z| < r$

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \ell(N) z^N, \tag{23}$$

Insightful fathers.

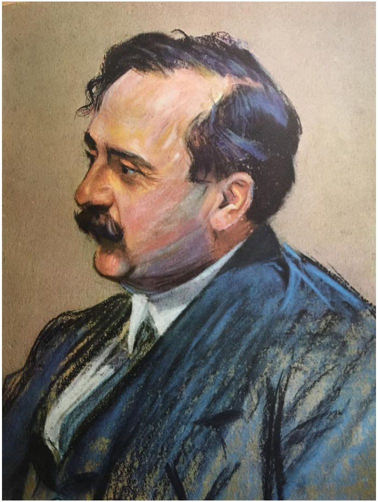


Figure: Jacques Hadamard and Paul Montel.

Local domains: morphism properties.

Corollary (of Theorem A)

Let $S, T \in$ *Dom*$^{\mathrm{loc}}(\mathrm{Li})$, *then*

$$S \sqcup\!\sqcup\, T \in Dom^{\mathrm{loc}}(\mathrm{Li}), \pi_X(\pi_Y(S) \,\text{⧢}\, \pi_Y(T)) \in Dom^{\mathrm{loc}}(\mathrm{Li})$$

and for all $N \geq 0$,

$$\begin{aligned} \mathrm{Li}_{S \sqcup\!\sqcup\, T} &= \mathrm{Li}_S \, \mathrm{Li}_T; \quad \mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)}, & (24) \\ \mathrm{H}_{\pi_Y(S) \,\text{⧢}\, \pi_Y(T)}(N) &= \mathrm{H}_{\pi_Y(S)}(N) \mathrm{H}_{\pi_Y(T)}(N). & (25) \\ \frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} &= \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \,\text{⧢}\, \pi_Y(T))}(z)}{1-z}. & (26) \end{aligned}$$

Continuing the ladder

$$
\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}
$$

We have, after a theorem by Leopold Kronecker,

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \left\{\frac{P}{Q}\right\}_{\substack{P,Q\in\mathbb{C}[x] \\ Q(0)\neq 0}} \tag{27}$$

On the right: freeness without monodromy.

Theorem (Deneufchâtel, GHED,Minh & Solomon, 2011 [12])

Let $(\mathcal{A},\partial)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $\partial(\mathcal{C})\subset\mathcal{C}$). We suppose that $k=\ker(\partial)$ and that $S\in\mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S)=MS\ ;\ \langle S\mid 1\rangle=1 \text{ with } M=\sum_{x\in X}u_x x\in\mathcal{C}\langle\langle X\rangle\rangle \tag{28}$$

(i.e. M is a homogeneous series of degree 1)
The following conditions are equivalent :

1. *The family $(\langle S\mid w\rangle)_{w\in X^*}$ of coefficients of S is (linearly) free over $\mathcal{C}$.*
2. *The family of coefficients $(\langle S\mid x\rangle)_{x\in X\cup\{1_{X^*}\}}$ is (linearly) free over $\mathcal{C}$.*
3. *The family $(u_x)_{x\in X}$ is such that, for $f\in\mathcal{C}$ et $\alpha_x\in k$*

$$\partial(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0).$$

A useful property.

mathoverflow

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U}\to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations $(N\geq 1)$

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots\ldots \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f\in\mathcal{U}^* | (\forall u\in\mathcal{U}_{N+1})(f(u)=0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^*\diamond\mathcal{U}_q^*\subset\mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^*=\cup_{n\geq 1}\mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U},.,1_\mathcal{U})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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Left and then right: the arrow $\mathrm{Li}_\bullet^{(1)}$.

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ш (x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof (pictorial).

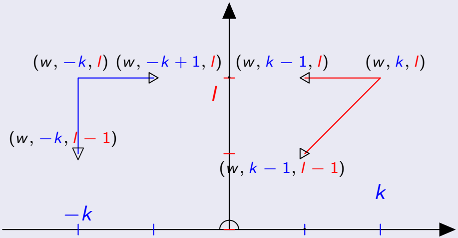

Figure: Rewriting mod $\mathcal{J}$ of $\{w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l}\}_{k\in\mathbb{Z}, l\in\mathbb{N}, w\in X^*}$.

Concluding remarks.

1. Extending the domain of polylogarithms to (some) rational series permits the projection of rational identities. Such as

$$(\alpha x)^* \sqcup\!\sqcup (\beta y)^* = (\alpha x + \beta y)^*$$

2. The theory developed here allows to pursue, for the Harmonic sums, this investigation such as

$$(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$$

3. We have, on the left, spaces equipped with Krull ultrametric convergence and a nice setting on the (topological) Magnus and Hausdorff groups. On the right, we have adapted domain theories with identities between polylogarithms and harmonic sums.

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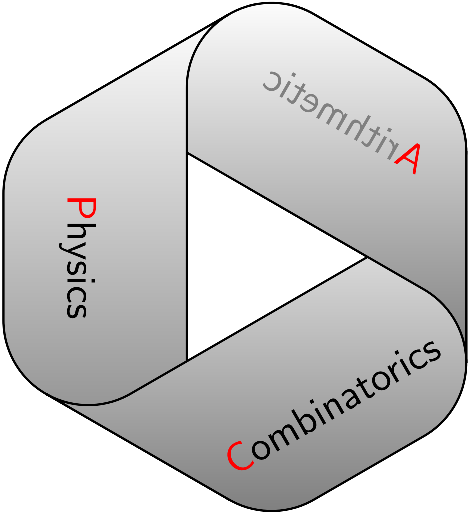

Figure: ... and a lot of (machine) computations.

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